Functions

Introduction

Function is one of those words with a mathematical meaning that is not the same as the everyday meaning. In everyday life, a function can be an important celebration, a role someone or something has, or the purpose for something. In mathematics and science, a function is a special and very important kind of relationship between variables. Discovering functional relationships between variables is what science is all about.

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Three Blind Men and an Elephant. There is a story about three blind men who encounter an elephant. The first comes up against one of the elephant's legs and says, "An elephant is like a tree." The second touches the side of the elephant and says, "No, no. An elephant is like a wall." The third blind man finds the elephant's trunk and claims, "You're both wrong. An elephant is like a big snake."

Each blind man has some notion of what an elephant is. The story doesn't tell whether they eventually resolve their differences and come to a proper understanding of elephants; we can only hope so. Our approach to functions will be similar: We will begin with three different views of functions. This, we hope, will lead to a fuller and more proper understanding of functions.

Tables

Many functions are displayed as tables. For example, consider this data table from the experiment *Mass vs. Number,* shown in Figure 1.

N **Number of Erasers**	***M*** **Mass (in gm)**
1	39 gm
2	79 gm
4	158 gm

Figure 1: Mass vs. Number *data table*

This table displays a functional relationship between the variables N and M. For each value of N we have a value of M; we say that the mass is a function of the number of erasers. That is, if we know what N is, then we can find M. This is the essence of a function: knowing one variable's value enables us to find the corresponding value of the other variable. In this example, the data has a distinctive pattern: within experimental error, doubling one variable causes the other to double. *N* and *M* are said to be (directly) proportional.

Many of the functions we encounter in everyday life come to us as tables. Stock tables, for example, can be thought of as functions: one variable is the company, the other is the closing price. If you know the company, you can look up the price. Almanacs are filled with tables of information, most of which can be thought of as functions: one variable is the name of the country, the other is the population, and so on. The sports pages are filled with tabular functions of team standings and individual statistics.

Many functions come to us first as tables, and some, like batting averages or almanac information, are normally given only as tables. Tables of numbers, however, can be very difficult to understand. Patterns in the data can go undetected—patterns that might help us better understand the function. One of the best ways to get a handle on patterns is to make a visual image of the data in the form of a graph.

Graphs

If a picture is worth a thousand words, a graph is worth a billion numbers. The graph of the *Mass vs. Number* data, seen in Figure 2, is a good example.

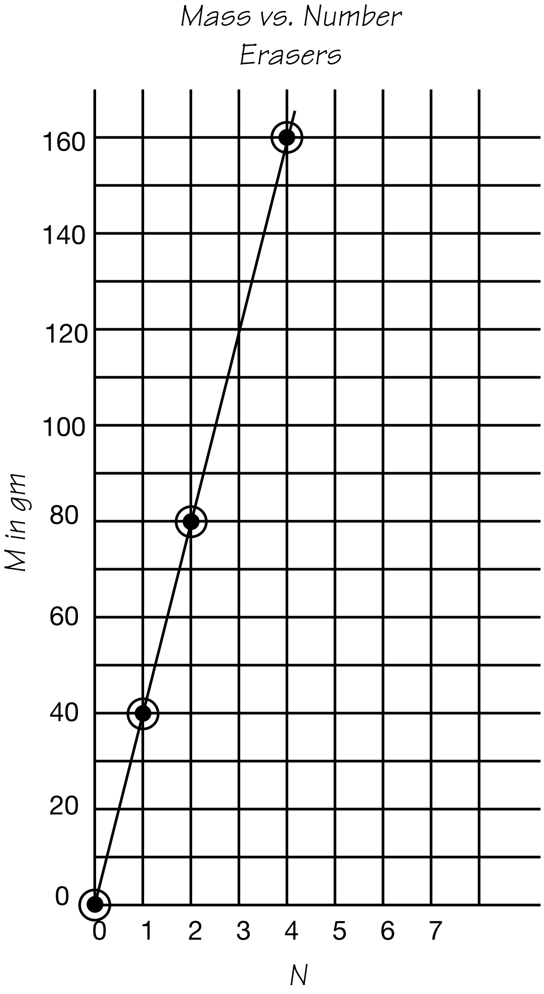

Figure 2: *Graph of* Mass vs. Number *data*

The key thing about this graph is that the data points lie on a straight line through (0,0). This confirms that *N* and *M* are proportional: we can fit a straight line through (0,0) if and only if the variables are proportional. Many other types of functions besides direct proportion are characterized by their graphs.

Often, experimental error obscures the nature of the relationship between the variables until the data is plotted. Like taking multiple measurements and averaging, graphing the data and fitting a curve can help control error. With error minimized, the true nature of the relationship between the variables may become clearer. Stock analysts graph their data to spot trends in order to make predictions (and money); such trends are unlikely to be noticed in the stock tables. Scientists graph their data almost as soon as they get it because the graph is so much more likely to be revealing than the raw data.

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Formulas

A more abstract way to consider a function is as a formula. In our *Mass vs. Number* example, we can exploit the fact that when variables are proportional, their ratio is constant. The value of this constant ratio is the slope of the best-fit line. So,

$$\frac{M}{N} = \frac{39.5 \text{ gm}}{1 \text{ eraser}} .$$

This can be rewritten to give a formula for *M* as a function of *N:*

$$M = \frac{39.5 \text{ gm}}{1 \text{ eraser}} \times N$$

Such formulas are very useful when they can be found. When we do manage to obtain a formula, it allows us to solve problems quickly and accurately. We can also use formulas to ascend to higher levels of abstraction. This movement to ever greater abstraction and generality is the driving force behind much of science and mathematics.

G Number of Generations	*A* Number of Ancestors
1	2
2	4
3	8
4	16

Figure 3: *Ancestors data table*

Often students are not able to understand formulas, but they are, nevertheless, able to continue a pattern in a data table or devise a rule that works. For example, the pattern in the table shown in Figure 3 is easy to extend. Five generations back, you have 32 ancestors: each further generation doubles the number of ancestors (ignoring the inevitable overlap). Extending patterns like this is a first step towards formulas. Later, the students may be able to give a rule for a data table. For example, consider the table in Figure 4, the number of fence posts needed for a given length of fence.

L Length of Fence in feet	*N* Number of Fence Posts
10	2
20	3
30	4
40	5

Figure 4: *Fence posts data table*

The rule may be stated: the number of fence posts is just one more than $\frac{1}{10}$ the number of feet in the fence.

$$N = \frac{L}{10} + 1$$

It is not much harder for us to express this sentence as a formula, but this type of expression may be confusing to a third-grader. Often, a rule stated in ordinary language is more accessible.

Formulas are the most powerful way of looking at functions but are often not appropriate for elementary school students. The great temptation is to drive on to formulas as quickly as possible. This sometimes leads to quick gains, but over the long run it is often problematic. Pushing formulas at children is like building a house of cards: students need to build a conceptual foundation by handling apparatus, gathering data, and graphing and analyzing it. Only after the students have developed an understanding of the relationship of the variables is it proper to distill that understanding into a formula. Extending patterns and figuring out rules are excellent alternatives for younger students moving towards higher levels of abstraction.

What Is a Function?

A **function** is a special kind of relationship between variables that can often be expressed as a data table, graph, or formula. One of the variables is the manipulated (or independent) variable; the other is the responding (or dependent) variable.

But not every relationship is a function. The main requirement for a relationship to be a function is that for each value of the manipulated variable, there is only one value of the responding variable. For example, suppose the manipulated variable is the edge length of a cube and the responding variable is the surface area. Then, for each given edge length there is exactly one surface area: if the edge is 3 cm, then the surface area is 54 sq cm, and so on. Or consider the manipulated variable to be the company and the responding variable to be the closing price: each stock has exactly one closing price each day. Notice that many stocks may have the same closing price; that's okay. The requirement says only that each value of the manipulated variable must have exactly one value of the responding variable. Values of the responding variable may repeat, and often do.

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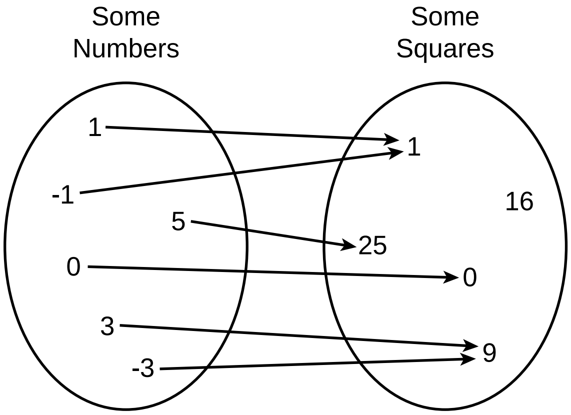

Figure 5: *Map of a squaring function*

One way to visualize this requirement is to think about functions as mappings. Figure 5 shows a map of a squaring function.

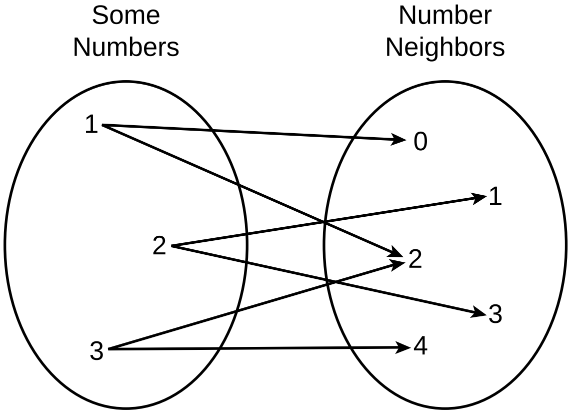

Figure 6: *Map of a relationship that is not a function*

Every number has exactly one square; this corresponds to the single arrow leaving each number on the left above. Notice that more than one arrow can end at a single number; the requirement is only that exactly one arrow leave each of the starting numbers. The mapping in Figure 6 is not a function because two arrows leave each number. To paraphrase Justice Potter Stewart, we may not know exactly what a function is, but we know one when we see it.

But you may object, saying that when we do experiments we often have several different values of the responding variable for each value of the manipulated variable. Isn't there a functional relationship between variables like the drop height and the bounce height in *The Bouncing Ball?* In a word, yes. But with real data, things get more complicated. Experimental error creeps in; there are other uncontrolled variables; there is uncertainty inherent in all our measurements. One useful way to think about the situation is to suppose there is some "true" value of the responding variable that we cannot measure exactly. So, instead we measure the responding variable several times and then take the average as our best estimate of the true value.[1] Then we have a function: for each value of the manipulated variable there is exactly one *true* value of the responding variable. The only trouble is, in most real-world situations, we usually don't know what that true value is.

Mathematicians have precise and abstract definitions of function. We could go on and on describing more precisely the requirements a relationship must have to be a function. We could spell out the meanings of technical terms having to do with functions—terms like *domain, range, one-to-one, many-to-one,* and so on. For most purposes, however, thinking about functions in the terms outlined above is enough.

Functions in the Classroom

The concept of a function is a powerful one for organizing and extending mathematical ideas. From time to time, we have activities that deal with functions, for example, *Function Machines* in second grade. The best approach to functions, however, is just doing the *Math Trailblazers* lessons. As your students do TIMS Laboratory Investigations and other activities, they will move naturally from the apparatus to the data table, graph, and questions. Much of the data analysis is designed to help the children see how changes in one of these correspond to changes in the others. Thus, in each experiment the students will deal with specific functions in several guises and will gain facility in moving between different representations of functions. This is the best possible preparation for the explicit study of functions later in high school and beyond.

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[1] See the TIMS Tutor: *Averages.*