The Concept of Length

Units

In dealing with scientific problems, we must often know such things as the location of an object, the distance between objects, how tall or wide an object is, or how fast it is moving. Central to all these ideas is the concept of length. In this tutor, we shall examine the various disguises in which length can appear.

First, however, we must note the messy problem of units. The kinds of units chosen for length were for a long time quite arbitrary. The cubit, used by several ancient civilizations, was the length of a forearm between the tip of the middle finger and the elbow. The fathom was the width of a Viking sailor's embrace. (Fathom that!)

The foot was a convenient length defined as the length of a person's foot. Of course, everyone has a different-sized foot, so if we want to use, say, the king's foot as a standard, we will have to mark it permanently; we cannot very well lug the king around! Since the king is the ruler of the land, it was both rational and proper to call this marker a ruler as well. To show students the chaos caused by not having a standard length, have them measure the length of their desks in cubits or the width of the room in feet with each child using his or her own body measurements. You will have as many different values as measurers.

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Nonstandard Units

At first, we have the children measure length using nonstandard units, often links. A link is shown in Figure 1 along with a chain of links. The measurement of the object shown is $4\frac{1}{2}$ links. The reasons we go to links are: (1) to make counting the unit length easy and (2) to keep the numbers manageable. Rather than dealing with hundreds of cm, we can count 50 links. And to count a link is easy since they are so large.

By alternating link colors (2 reds, 2 whites, 2 reds, etc.), one can make the counting even easier by skip counting by 2s. If you want to have the children skip count by 5s, then have a chain with 5 blues, 5 yellows, 5 blues, etc.

With links, it is also easy to round off to $\frac{1}{2}$ link as shown in Figure 1. If the edge we are measuring is near the middle, then the length is $4\frac{1}{2}$ links. If shorter, then the length is 4 links; if longer, the length is 5 links.

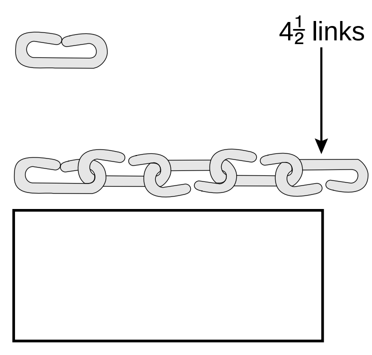

Figure 1: *Links*

> *. . . the French scientific community at the time of the French Revolution (c. 1790) chose as the standard unit of length a distance that they called a meter. It was chosen so that 10^7 (10 million) metersticks laid end to end would just fit between the North Pole and the equator . . .*

The Metric System

Of course, links are not a commonly used standard of measure. In the United States, it is feet. Unfortunately, many objects are smaller than a foot. This means one must subdivide the standard unit. For some miserable reason, the smaller unit, the inch, is $\frac{1}{12}$ of a foot. Who likes to divide by 12? So, the French scientific community at the time of the French Revolution (c. 1790) chose as the standard unit of length a distance that they called a meter. It was chosen so that 10^7 (10 million) metersticks laid end to end would just fit between the North Pole and the equator, as shown in Figure 2. A platinum-iridium rod was constructed with two marks a meter apart and stored in a vault near Paris. Every meterstick, albeit indirectly, comes from this standard. Having defined the meter, the French were smart enough to define all subsequent subdivisions of the meter as integral powers of ten. The foot is divided into $\frac{1}{12}$s and $\frac{1}{48}$s, and other equally horrible numbers; for the meter, the divisions are $\frac{1}{10}$s, $\frac{1}{100}$s, and $\frac{1}{1000}$s. We shall now explore this point further.

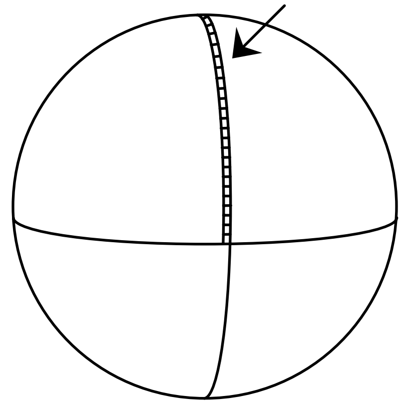

Figure 2: *Metersticks from the North Pole to the equator*

For measuring ordinary objects in the classroom, the meter is divided into three other units; the millimeter, the centimeter, and the decimeter (although the latter is rarely used). Without being specific as to the size of each unit, we can order them using the greater than (>) or less than (<) sign. Starting with the smallest, we have:

$$1 \text{ millimeter} < 1 \text{ centimeter} < 1 \text{ decimeter} < 1 \text{ meter}.$$

Reversing the order and starting with the largest, we have:

$$1 \text{ meter} > 1 \text{ decimeter} > 1 \text{ centimeter} > 1 \text{ millimeter}.$$

It is important that students know at least this much before going on to more exact relations.

The key to the subdivision of the meter is the prefix *milli.* Milli is related to the word mile. Mile was the distance it took a Roman soldier to step off 1000 paces, a pace being two steps. Since an average pace is approximately 5 feet (try it and see), a mile would be approximately 5000 feet. The crucial point is the number 1000 as related to the word mile. Milli is the prefix for "$\frac{1}{1000}$ of." Thus, a millimeter is one-thousandth of a meter. There are 1000 millimeters in a meter just as there are 1000 paces in a mile. In each case, we have a subunit that is one-thousandth of the main unit. The word mile is to remind us that there are 1000 paces in a mile. The word millimeter is to remind us that there are 1000 millimeters in a meter. One should try to picture this in one's mind. Millimeters are tiny; it takes a lot of them to make up any macroscopic length. (By macroscopic, we are referring to something one can see unaided versus microscopic where one would need a magnifying glass or microscope to see it.) On the other hand, because a meter is large, a millimeter is usually a fraction of most lengths one would measure in the lab. Thus, it is reasonable that a third-grader knows how to measure the length of his thumb as 3 cm or 30 mm, but not as 0.03 m.

How are mm, cm, dm, and m related? Let's start with the smallest and see how many mm are in a cm, a dm, and a meter.

1 centimeter contains 10 mm;
1 decimeter contains 100 mm;
1 meter contains 1000 mm.

We should be able to figure out, based on these relationships, how many centimeters are in a decimeter or in a meter. Here, however, the French have made it easy for us; the prefix for each word gives the answer away. *Centi* stands for 100th and *deci* stands for one-tenth. Thus one centimeter is one-hundredth of a meter; there are 100 cm in a meter. A decimeter is one-tenth of a meter; there are 10 dm in a meter. What this boils down to then is the following:

1 decimeter contains 10 cm;
1 meter contains 100 cm;
1 meter contains 10 dm.

Everything depends upon the size of the meter. Once that is fixed (by our rod in Paris), the sizes of all other metric units are determined.

Measuring Length

The simplest way to get started in the metric system is to count, using a meterstick, the number of mm or cm (and if they do not equal fractions, dm or meters) in a given length. Say we measure the width of a sheet of paper in cm. Then depending upon the accuracy of the meterstick (a cheap one could be off a bit) and the judgment of the student, one can see that there are between 21 and 22 cm across the page. If you stick to cm, then for students who are not yet comfortable with fractions or decimals, this is all you can say. As they begin to learn decimals, the children can determine the width as 21.6 cm. However, you can get still better accuracy even without decimals by going to mm instead of cm. Here, all one has to be able to do is count beyond 100. Thus, the width is 216 mm. In fact, one of the neat things about the metric system is that you can always choose a set of units to obtain almost any accuracy you want without going to fractions or decimals. On the other hand, you can purposely choose units that will give decimal or fractional answers. As we just saw, in cm units, the width of the page is a decimal, 21.6 cm. We could have asked for the width in meters. Since the width is less than a meter, we are dealing with fractions. In this case, the width is 0.216 meters or roughly $\frac{1}{5}$ of a meter. Clearly, there is great potential in the metric system for teaching math and linking this to scientific measurement.

With regard to addition, when adding numbers they must always have the same units. For example:

(a) 5 cm + 6 m = ? This is a "no-no"; the units are mixed.

(b) 5 cm + 600 cm = 605 cm. This is okay—we are adding the same units.

(c) 8 mm + 50 cm = ? We should convert the cm to mm and get:

(d) 8 mm + 500 mm = 508 mm.

Thus, if we ask a student to measure the length of his arm by separately measuring his hand (say in mm), his lower arm (say in cm), his upper arm (say in decimeters), and then adding them, he will first have to convert to a set of consistent units that he or she can handle. If you do not choose to have the students work with fractions, the students can change the units to millimeters as shown above. If you want to give the students a chance to work on decimal fractions, they can change the units to centimeters.

$$.8 + 50 \text{ cm} = 50.8 \text{ cm}$$

This brings us to another point—how to use a ruler. At first it seems quite apparent: just place the end of the ruler at the end of the object and read the length directly (Figure 3). However, a better test of whether students really understand how to use a ruler as well as a test of their ability to subtract is to place the object in the center of the ruler. Clearly, the length of the object should not depend upon its position vis-à-vis the ruler, but we have found that many young people (and even a few at our university) have trouble understanding how to find the length in the latter case.

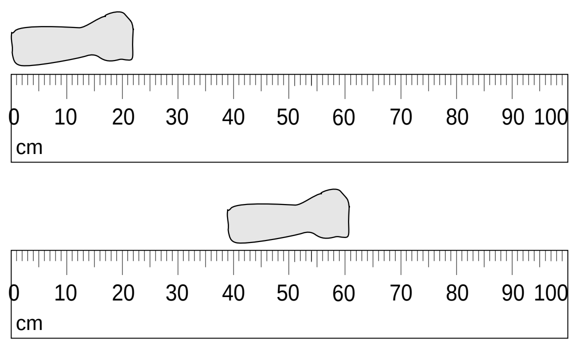

Figure 3: *How to use a ruler*

Since we do not always have a ruler handy, a few "natural" rulers might be fun to discuss and use. For example, say the length of a person's upper thumb from knuckle to tip is generally about 3 cm while his spread-out fingers span about 20 cm, as shown in Figure 4. Either can now be used to measure the length of an object. Of course, there is the foot, a convenient measure for stepping off distances. This person's foot, without his shoes, is 23 cm or about 9 inches. Anyway, you should have the children measure a few objects using natural rulers and have them compare their results with that of a meterstick.

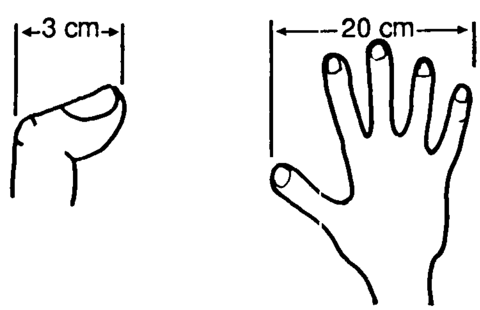

Figure 4: *Natural rulers*