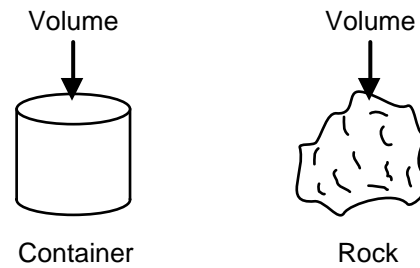


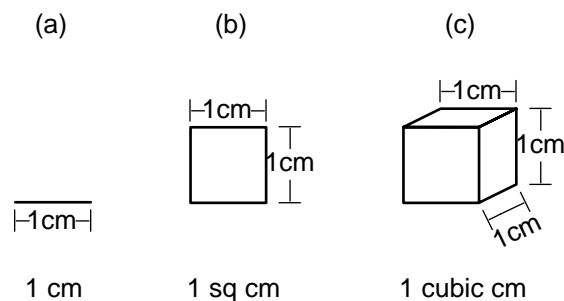
# The Concept of Volume

## Defining Volume



**Figure 1:** Two meanings of volume

Figure 1 shows a container and a rock. The space that the container surrounds (and is occupied by air) and the space that the rock takes up (and is occupied by elements such as oxygen, silicon, and aluminum) are both called **volume**.<sup>1</sup> The concept of volume is tricky. Two objects (like our container and rock) might occupy the same volume but might contain totally different amounts of matter. Children often confuse the amount of matter, which we call mass, with the space occupied, which we now know is volume. Thus children tell us that a “heavy” object has more volume than a “light” object even though the latter may actually occupy more space. Indeed, volume is so oversimplified in the elementary schools that many eighth-graders we asked thought of volume as  $\text{length} \times \text{width} \times \text{height}$ , no matter what the shape of the object. Others told us that volume was length squared. Misconceptions such as these are a result of a curriculum that emphasizes memorization of formulas without attention to the conceptual foundations of volume.



**Figure 2:** Units of length, area, and volume

<sup>1</sup>Sometimes, the space inside a container is called the capacity of the container. For simplicity, we prefer to use one term, volume, to denote both “space inside” and “space occupied.”

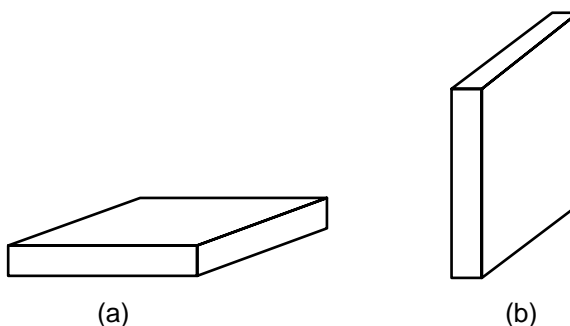
## Units

As with length and area, if we wish to measure volume, we need to decide on a unit of measure. In the metric system, the metric unit of length is the centimeter (Figure 2a), and the unit of area is the square centimeter (the extent of the plane surface that is bounded by a square 1 cm on a side) (Figure 2b), so it is not unreasonable that we take as our unit of volume the space occupied within a cube that is 1 cm on a side (Figure 2c). The volume occupied by such a cube is defined as 1 cubic centimeter whether that volume is occupied by a solid object (Figure 3a) or by empty space (Figure 3b).



**Figure 3:** Full and empty cubic centimeter

Unlike area, it is very hard to divide an object up and count cubic cm. We can't trace the volume the way we trace areas on square cm paper. In theory we could slice it up into 1 cubic cm pieces, but this process will destroy the object. Therefore, learning to understand and measure volume can be more difficult than understanding and measuring area.



**Figure 4:** Flat vs. tall shapes

In everyday life, we are accustomed to a variety of other units of volume measure. Gallons, quarts, pints, and fluid ounces are one set of units, usually used to measure the capacity of a container. Cubic feet and cubic inches are also frequently used. For example, in the United States, air conditioners are often rated on the volume of space they can cool, measured in cubic feet. The remainder of this tutor uses metric units of measure, but many of the same underlying ideas apply to any system of measure.

## Problem of Dimensionality

Another great difficulty in understanding volume is that the concept deals with three dimensions. As Piaget pointed out, it is much easier, and therefore usual, for a child to focus on one dimension. Children will decide that a tall object has lots of volume because they only focus on the height and fail to take into account the other two dimensions to make a proper estimate. In Figure 4, the two objects have the same volume, but because (a) is flat and (b) is upright, young children will tell you that (b) has the greater volume.

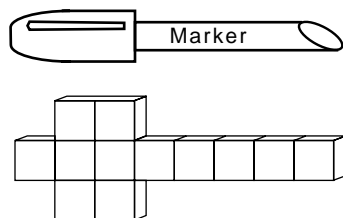
Volume is an extremely important scientific variable. The way it is related to area and to mass and the manner in which it may change with time are all intrinsic to every area of scientific investigation. It is well worth our time to do a good job on volume.

## Measuring Volume—Early Activities

Kindergarten students explore volume by filling containers with rice, water, pasta, sand, or beans. Another way to deal with volume in the primary grades is to have the children make figures out of a set of cubes. In first and second grade, we usually use connecting cubes that are about  $\frac{3}{4}$ -inch on a side. At the end of second grade, we start using standard centimeter linking cubes. For example, you can give each child 10 cubes and ask him or her to make a figure whose volume is 10 cubic units. You will get a variety of shapes, all of which have the same volume. This will begin to impress upon the children the idea that many different shapes can have the same volume.

Another centimeter cube activity to build the children's understanding at this level involves building shapes with different volumes. Give each child a few (3–10) centimeter cubes. Have each child make a shape with his or her centimeter cubes. Then have the children sort themselves into groups according to the volume of their shapes.

You can bring in some simple solid shapes, like a piece of chalk, a match box, a pile of washers, etc., and have the children make figures out of cubic units that approximate the volume (size and shape) of these objects. In this way, they can estimate the volume of the original object by keeping track of the number of cubic cm they used. An example is given in Figure 5 of a marking pen and cubic cms linked together to make a shape of approximately the same volume.



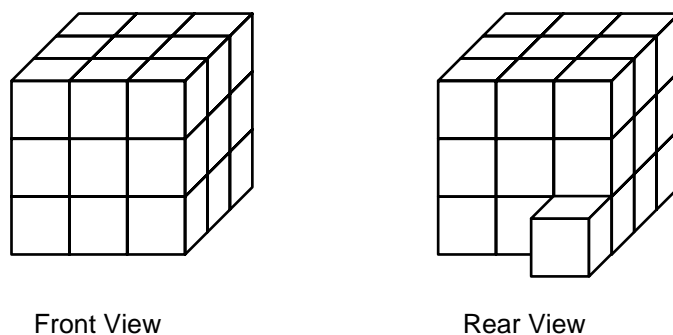
**Figure 5:** A marker and a cm cube model



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might be some unexpected cubes hidden behind the model in Figure 6c (see Figure 7).



**Figure 7:** Front and rear views of Figure 6c

For example, in Figure 7 we see what appears to be a  $3 \times 3 \times 3$  cm cube from the front, but when we view the model from the rear, we see there is an “extra” cube stuck on.

Finally, and hardest of all, you can ask the children to try to draw a figure with a given number of cubic centimeters. It is very hard for anyone to draw a cube, and a figure with several cubic centimeters is harder still. Nevertheless, it is worth a try since doing so will help them improve their spatial perception and will force them to think in three dimensions.

Let’s review the four steps we have just described:

1. Make figures out of a given number of centimeter cubes.
2. Count cubic centimeters in a cube model.
3. Count cubic centimeters in the drawing of a cube model—usually prisms.
4. Draw a figure with a given number of cubic centimeters.

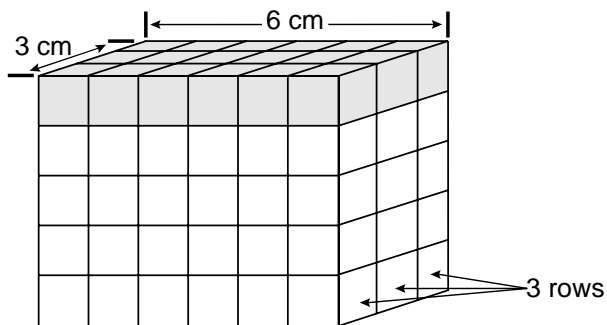
This discussion covers volumes of objects made from cubes. The most important tool, however, for determining volume will be the graduated cylinder which we will discuss in the section after next. But first we want to talk about how the volume of prisms can be calculated.

## Calculating Volume—An Upper Grade Exercise

The volume of a rectangular prism made of cubic centimeters can be found using multiplication. The number of cubic centimeters in the top layer is just the product of  $3 \times 6$  since there are 3 rows of 6 cubic cm (see Figure 8). Thus, in each layer there are 18 cc. Because there are 5 layers, the total number of cubic centimeters is  $5 \times 18$  cc = 90 cc. As often written in math books, this type of counting is expressed as

$$V = l \times w \times h.$$

You should interpret this as the number of cubic cm in the top layer (given by the value of  $l \times w$ ) times the number of layers (given by the value of  $h$ ).



**Figure 8:** A rectangular prism

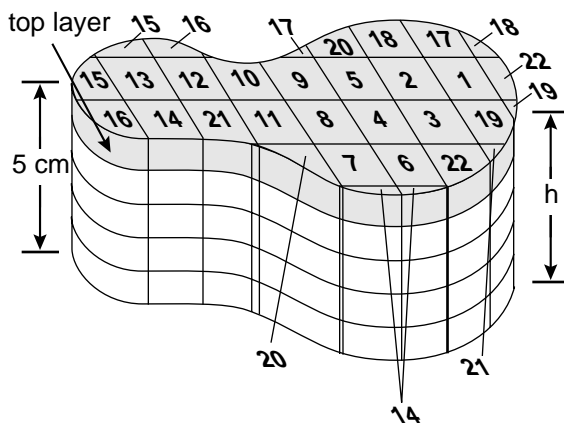
If the top layer is not rectangular but the figure is a right solid (sides perpendicular to top and bottom), then we can still find the volume by the above technique. The formula  $length \times width \times height$  will no longer work since the layers are not rectangular. However, since each horizontal slice has the same shape (see Figure 9), all we have to do is find the number of cubic cm in the top layer and multiply this by the number of layers. To find the number of cubic centimeters in the top layer, we have to find the number of square centimeters in the top surface, since each square centimeter of the surface is attached to a cubic centimeter in the top layer. Thus, if in Figure 9 by counting square cm we find that there are 22 sq cm on the top surface, then there must be 22 cubic cm in the top layer and in each subsequent layer. The total volume, then, is:

$$\frac{22 \text{ cubic cm}}{1 \text{ layer}} \times 5 \text{ layers} = 110 \text{ cubic cm}$$

for the object shown in Figure 9. As a general formula we have:

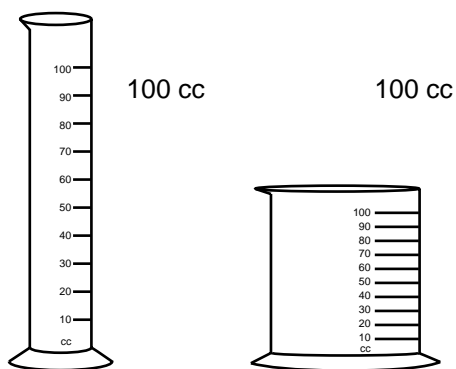
$$V = A \times h,$$

where the area  $A$  of the top tells us the number of cc in the top layer, and the value of  $h$  tells us the number of layers. The children should not just memorize each formula. They should understand what is behind the formulas.



**Figure 9:** A right solid

When using a formula like  $l \times w \times h$ , the old bugaboo of units reappears. Since  $l$ ,  $w$ , and  $h$  are all measured in centimeters, it is tempting to say that the units of volume are centimeters cubed written as  $(\text{cm})^3$ . And indeed this is what is often done in scientific texts. Yes, it's technically correct to write  $(\text{cm})^3$  or  $\text{cm}^3$ , but again it can be misleading for children just as "centimeter squared" can be misleading for area. If we say a volume is 7 centimeters cubed, is that  $(7 \text{ cm})^3 = 343 \text{ cc}$ ? We can avoid this confusion by writing what we mean; that the volume is 7 cubic centimeters or 7 cc.

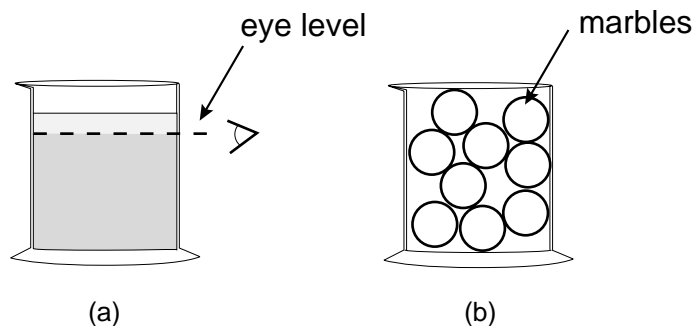


**Figure 10:** Graduated cylinders

Clearly, the ability to calculate volume is rather limited to certain special shapes. An irregularly shaped container or object will require a different approach.

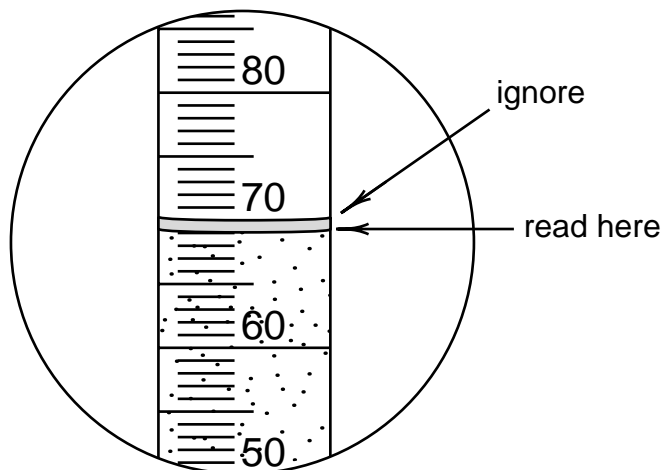
## A Volume Measurer: The Graduated Cylinder

We have seen that a ruler which is calibrated in cm can be used to measure length. There is no comparably simple device for measuring area, but there is one for measuring volume. It is the graduated cylinder calibrated in cubic centimeters. Two graduated cylinders are shown in Figure 10. They can be made of glass or Pyrex, both of which are breakable, or plastic, which is not. The cylinder most suitable for classroom use would be calibrated in 1 cc, 5 cc, or 10 cc divisions and have a capacity of 100 to 150 cc. When filled with a liquid (usually water) or a fluid substance like sand or salt, one can read the volume of the material off the side of the graduated cylinder. One cannot use this device to directly measure the volume of a number of marbles since the marbles piled in the cylinder will leave an unknown volume of air spaces between them. This is illustrated in Figure 11b. As we shall see in the next section, we can find the volume of solid objects like marbles by the method of displacement. The rest of this section will discuss using the graduated cylinder for finding the volume of liquid, particularly water.



**Figure 11:** Liquid and marbles in graduated cylinder

To read the volume of water in a graduated cylinder, your eyes must be level with the liquid, as shown in Figure 11a. Since water is pulled up at the sides of the cylinder into a curved surface called a meniscus, one must measure the water level at the center of the cylinder. This is done by using the lower of the two lines that one sees (Figure 12) when looking at the water from the side. The top line is due to the pulled up water and should be ignored. (This phenomenon tends to be more pronounced when using glass rather than plastic graduated cylinders.)



**Figure 12:** *The meniscus*

One must be careful about what one puts into a graduated cylinder. A tall, narrow one will generally have 1 cc divisions and thus can be used for accurate measurements. The trouble is, that because it is narrow, only small objects can be used. Many a time a student has misjudged the size of an object and found it stuck in the narrow cylinder never to come out again.

Chalk off one cylinder. A good general size that we like is one about 4 cm in diameter with a 150 cc capacity and made of plastic. The divisions are usually 5 or 10 cc. We shall discuss in the next section how to use a “big” graduated cylinder to measure the volume of small objects.

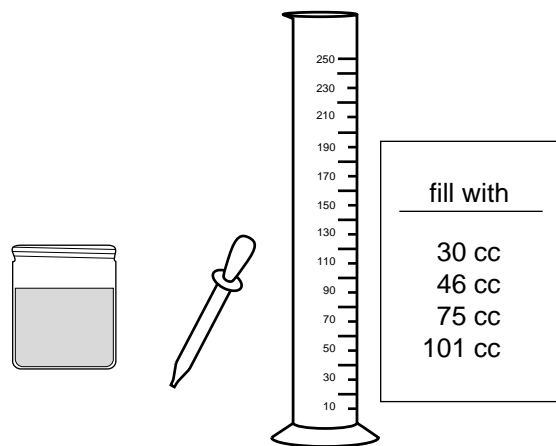
Many graduated cylinders that you purchase will have several different scales along the sides. This is due to the diversity of units for liquid measurements. Many cylinders will have a scale for fluid ounces and another for milliliters. The metric unit of volume is the milliliter, which is defined to be 1 cubic cm:

$$1 \text{ ml} = 1 \text{ cc.}$$

The liter is often encountered in daily life (for example, soda bottles are often 1 or 2 liters). It is exactly 1000 ml.

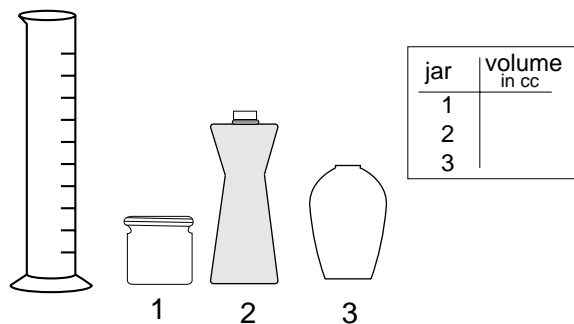
One of the first exercises the children can do with a graduated cylinder is simply to fill the cylinder to a specified level—10 cc, for example. An eyedropper is handy for getting the volume exactly right, since by pouring

one usually overshoots or undershoots the mark. Whatever the divisions of the cylinder are, choose some volumes that fall right on a major division and some that fall between divisions, where the children will have to interpolate. An example is shown in Figure 13 for a cylinder with 10 cc divisions.



**Figure 13:** A graduated cylinder exercise

Once the second- or third-graders are good at reading the scale, they can use the graduated cylinder to find the capacity or volume of a set of three jars (Figure 14). You should build up a collection of jars of all shapes and volumes, from small baby food jars, through peanut butter containers, to large coffee jars. Exotic shapes are nice. To find the volume of a jar, one can either fill the jar to the brim and then keep pouring the water into the graduated cylinder, or one can keep filling the graduated cylinder and pour the water into the jar until it is filled. Either way, the children have to keep track of the number of times the graduated cylinder is filled and the total volume of water accumulated this way. This activity appears as a lesson called *Fill 'er Up* in third grade.



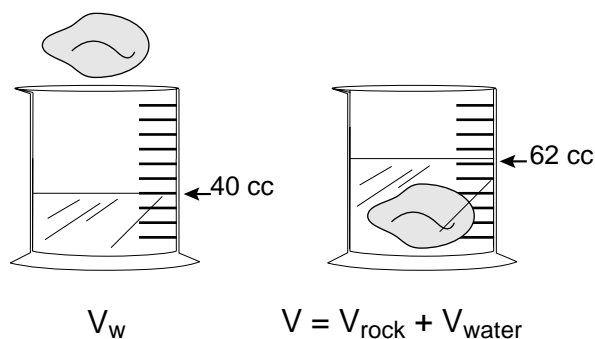
**Figure 14:** A capacity exercise

## Volume Measurement by Displacement

How would you measure the volume of a small rock? This problem is dealt with in lessons on volume beginning in Grade 2. The technique is illustrated in Figure 15. First, you fill the graduated cylinder with a convenient amount of water, for example, 40 cc rather than 43 cc. Then you place the rock in the graduated cylinder without losing any water (there must be enough water in the cylinder initially to cover the object). You then read the new volume  $V$ . Since the volume of water  $V_{\text{water}}$  stays constant, the volume  $V$  is due to the water plus the rock. The rock displaces, or pushes aside, its volume in water and the water level rises. Thus, we have:

$$V_{\text{rock}} = V - V_{\text{water}}$$

For example, in Figure 15 the volume of the rock is 22 cc. Note that subtraction is easier if you start with a multiple of ten for the volume of water. The technique works for any solid object no matter what its shape. To start with, then, the children should be asked to find the volume of a wide variety of objects, some spheres, cubes, rocks, coins, washers, etc.

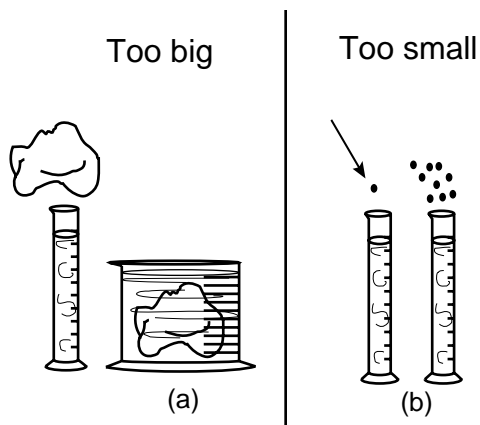


**Figure 15:** Volume by displacement

But what if the object is too big to fit into the graduated cylinder? If a bigger graduated cylinder isn't handy, then you can use your graduated cylinder to calibrate a large jar, and then away you go. One word of warning: When you place a large object in a graduated cylinder or jar, a considerable amount of water may splash out, even though you are very careful. You can get around this by placing the object in the graduated cylinder first and then pouring in a known amount of water  $V_w$ , and then read  $V = V_{\text{rock}} + V_{\text{water}}$  off the scale. An alternative method is to find a large container and fill it with water to the brim. When the object is carefully placed in the container, it overflows, and the volume of the overflow liquid is equal to the volume of the object. If you catch all the overflow liquid in a second container, you can then use a graduated cylinder to find the volume of that liquid.

Finding the volume of small objects is also a problem, since the object may be so small you can't see the water level rise. Of course, you can always use a smaller graduated cylinder, one with 1 cc divisions instead of 5 or 10 cc divisions. The trouble is that sometimes even 1 cc per division is too large. The only way out is to measure the volume of several of the small

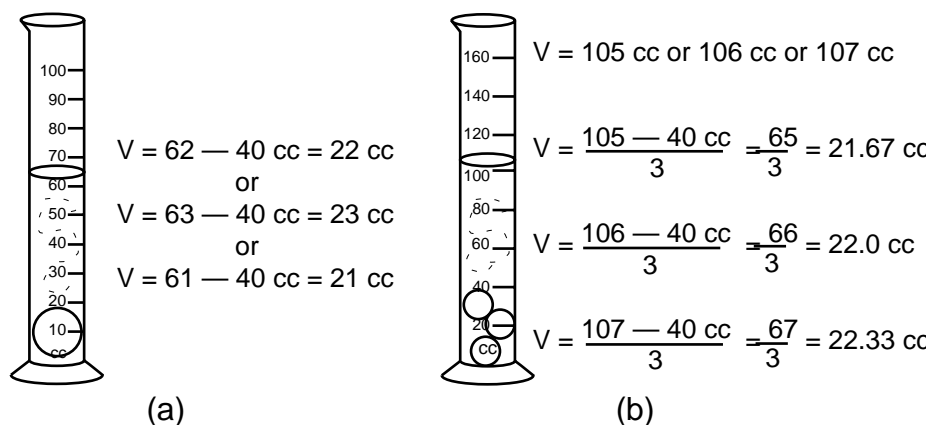
objects at once (Figure 16b). (The volume of the several objects are identical.) For example, suppose you have 10 identical objects. To find the volume of one object, subtract the volume of the water and divide by 10. In this fashion, the children can find the volume of small washers, paper clips, pins, etc. Using 10 objects is a good idea since it is easy to divide by 10.



**Figure 16:** Finding volumes of large or small objects

## Obtaining Accurate Results

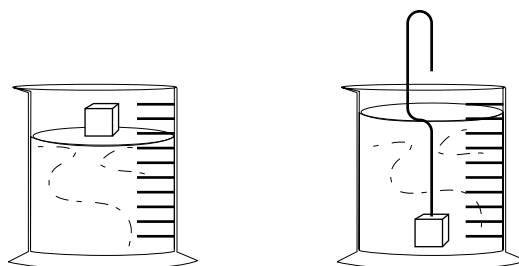
A few words about accuracy are appropriate here. Most of the time you have to interpolate between divisions in order to read the volume. This usually leads to a reading error of about 20% of the value between the scale marks. Thus, if your graduated cylinder has 10 cc divisions, you might expect an error (i.e., children will get readings that differ) by up to 2 cc. This is illustrated in Figure 17a. If the object has raised the water level several divisions, then this 20% per division uncertainty is not a serious problem. For example, our rock changed the water level from 40 cc to 62 cc. If a reading error is up to 2 cc, then three children who read the same graduated cylinder might read 62 cc, but possibly 61 cc or 63 cc (if they are careful; maybe even more if they are not). But the volume of the rock is 62 cc – 40 cc = 22 cc for one child, but 23 cc for another, and 21 cc for the third. Thus we have a spread in reading of about 2 parts in 22 or 10%. A bigger rock, with a volume of, say, 46 cc would still have the same reading error of 2 cc, but its volume error would only be 2 parts in 46 or about 5%.



**Figure 17:** Measurement error

A good rule of thumb is that your reading will not be accurate unless the object raises the water level by more than one division. There are two ways to achieve this. One, use a narrow graduated cylinder, or two, use several identical objects if they are available. In the case of the latter, you find the volume of one object by dividing by the number of identical objects, as we did above for very small objects. For example, three objects would still produce a reading error of 2 cc but a volume error of only  $2 \text{ cc}/3 = 0.67 \text{ cc}$ . This is illustrated in Figure 17b where three marbles give a volume spread of only 0.67 cc in 22 cc or 3%, compared to 10% for one marble. Of course, this would not work for finding the volume of a rock, since it would be difficult to find three identical rocks.

If the object floats, you have to push it under in order to measure its volume. In this situation, what is important is how you push it under. If you use your finger, then what you measure is the volume of the object plus the volume of your submerged finger. Since the volume of your finger may be comparable to the volume of the object, this is clearly not a good idea. What you need is a pusher whose volume is much less than the volume of the object. A straightened paper clip or a pin will do (Figure 18).

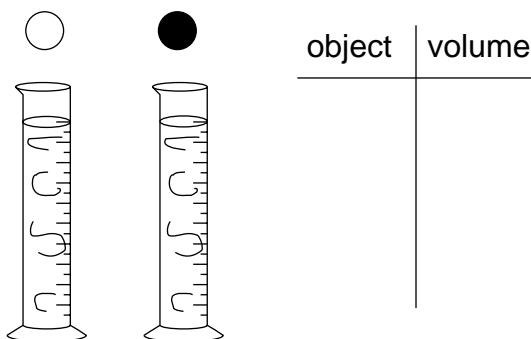


**Figure 18:** Finding the volume of something that floats

## Two Misconceptions

Early on, the children should come to grips with two important ideas concerning volume:

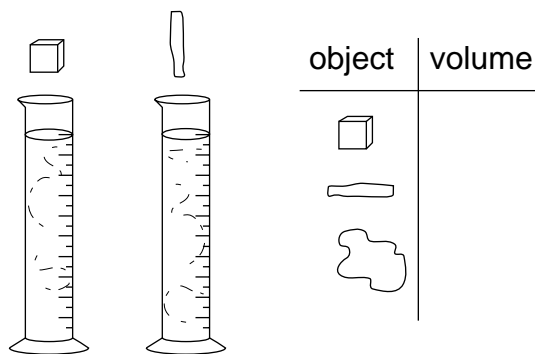
1. The volume of an object is independent of the material it is made of.
2. The volume of an object does not change when its shape changes.<sup>2</sup>



**Figure 19:** Volume vs. material

<sup>2</sup>Usually. See later discussion.

As to the first point, you can study spheres of the same volume but made of a wide variety of materials (steel, lucite, glass, even wood). The children should discover that their volumes are the same. As illustrated in Figure 19, students can fill two identical cylinders to the same level with water, carefully place both objects in, and see that the volume displaced is the same. Initially, many children will say that the heavier object has more volume and that the water will go up higher in its graduated cylinder. They are confusing mass and volume. If you take a clay cube and mash it into a thin disk, many children will say that the disk has less volume than the cube. Here they confuse one dimension, thinness, with volume: they mistakenly assume all thin objects have a small volume. Again, using the graduated cylinder the child can see that the volumes are the same (Figure 20). They can make all kinds of shapes out of a single piece of clay and determine that they all have the same volume. Indeed, the only way that they can change its volume is to tear off a piece.



**Figure 20:** *Volume vs. shape*